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HAAR WAVELET BASED COMPUTATIONALLY EFFICIENT OPTIMIZATION OF LINEAR TIME VARYING SYSTEMS

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ABSTRACT

Optimization using existing methods is computationally costlier task for complex control systems such as a high order system with time varying parameters. Various authors' solved optimization problems using different types of signal transforms followed recently using Haar wavelet transform which proved to be an excellent mathematical tool in signal processing. In this domain, the pioneering work has been done by Chen and Hsiao using computationally inefficient recursive Haar operational matrices. In this paper optimization of linear time variant systems has been done using computationally efficient non-recursive Haar backward integral operational matrix. The computational time savings, with respect to increase in resolution, obtained in the proposed method vis-à-vis Hsiao method are computed using MATLAB 7.6.0.324 (R2008a) thereby establishing computational efficiency of the proposed method.

KEYWORDS: Haar Wavelets, LQR, Optimal Control, Time-Varying Systems

INTRODUCTION

Optimal control problems with linear time varying systems are much more difficult to solve in comparison with linear time invariant systems. When the order of system is sufficiently high, it is almost impossible to find out the optimal solution using analytical methods. Various types of numerical methods can circumvent the problem. Since numerical methods involve lot of calculations and time, various signal transforms based methods are employed [1-4]. One such popular signal transform is Haar wavelet forming the basis of operational approach which has been extensively employed for solving most popular LQR optimal control problems. The approach, pioneered by Hsiao [5], is important because of nice properties of Haar wavelet. In this approach, various recursive Haar operational matrices are derived and applied for solving differential system equations [5,6]. The use of recursive operational matrices is computationally expensive task. Therefore, a modified method has been proposed in this paper where backward integration matrix has been replaced by equivalent non-recursive formulation, pioneered in [7], resulting in computational savings.

REVIEW OF HAAR WAVELET

The orthogonal set of Haar wavelet $h_i(t)$ is a group of square waves with magnitude of +1 and -1 in certain intervals and zero elsewhere. The first curve is $h_0(t)$ known as scaling function or father wavelet. The second curve $h_1(t)$ is the fundamental square wave which is also known as mother wavelet. By binary dilation and translation of $h_1(t)$, one can obtain a complete orthogonal basis for the space of integrable functions over $t \in R$. In other words $\{h_0(t), h(2^j t - k), j \in Z, k \in Z\}$ can span $\mathcal{L}^2(R)$, where $h_0(t) = 1$, j is the dilation parameter and k is the translation parameter. For practical applications a signal is usually obtained from initial time up to some finite time. Thus, without loss of generality, the signal can be normalized to the time interval $t \in [0,1]$.under the assumption, the signal in the space $L^2[0,1]$ can be represented by the orthogonal basis $\{h_n, n \in Z^+\}$, Where

$$h_0(t) = 1 \quad (1)$$

$$h_n(t) = h_1(2^j + k), \quad (2)$$

$$n = 2^j + k, j \geq 0, 0 \leq k \leq 2^j \quad (3)$$

OPERATIONAL APPROACH AND OPERATIONAL MATRICES

Haar wavelet based Operational approach simplifies analysis of complex systems by converting its integral differential system equation into linear matrix algebraic equations in Haar domain. Haar domain means that all the variables are transformed using Haar transform and all associated mathematical operations are transformed using Haar operational matrices. For example Integral operation is reduced to linear algebraic equations with the help of integral operational matrix [8].

$$\int_0^t h_m(\tau)\, d\tau = P_m h_m(t) \quad (4)$$

Other operational matrices are multiplication matrix M, connection coefficient etc., summarized in Table 1

Table 1: Recursive Haar Operational Matrices

Operational Matrix	Recursive Formulation	Elements
Multiplication matrix $M_m(t)$	$\begin{bmatrix} M_{\frac{m}{2}}(t) & H_{\frac{m}{2}} \mathrm{diag}[h_b] \\ \mathrm{diag}[h_b] H_{\frac{m}{2}}^{-1} & \mathrm{diag}\left[H_{\frac{m}{2}}^{-1} h_a\right] \end{bmatrix}$	$M_1(t) = h_0(t)$ $h_a = [h_0\ h_1 \ldots\ldots h_{m/2-1}]^T$ $h_b = [h_{m/2}\ h_{m/2+1} \ldots\ldots h_{m-1}]^T$
Connection matrix C_m	$\begin{bmatrix} C_{\frac{m}{2}} & H_{\frac{m}{2}} \mathrm{diag}[c_b] \\ \mathrm{diag}[c_b] H_{\frac{m}{2}}^{-1} & \mathrm{diag}[c_a^T H_{\frac{m}{2}}] \end{bmatrix}$	$C_1 = c_0$ $c_a = [c_0\ c_1 \ldots\ldots c_{m/2-1}]$ $c_b = [c_{m/2} c_{m/2+1} \ldots\ldots c_{m-1}]$
Backward integration matrix S_m	$\frac{1}{2m}\begin{bmatrix} 2mS_{\frac{m}{2}} & -H_{\frac{m}{2}} \\ H_{\frac{m}{2}}^{-1} & 0 \end{bmatrix}$	$S_1 = -1/2$

All the matrices defined above are recursive matrices. Computing operational matrices through recursive formulation is a computationally costlier task. However, computational cost is expected to reduce if operational matrices are computed without recursion.

In the study of time varying system via Haar wavelets, it is generally essential to calculate $h_m(t)h_m^T$. The product of $h_m(t)h_m^T$ is called Multiplication matrix [5]. The multiplication matrix $M_m(t)$ [6] has been defined in Table I. With the help of this recursive formula multiplication matrix $M_m(t)$ for any order m can be derived. Where $M_{m/2}$ and $H_{m/2}$ are multiplication and Haar matrices for half resolution respectively.

In wavelet analysis for a system, all functions need to be transformed into Haar series. The multiplication of Haar wavelets also requires to be expanding into Haar series. Therefore a connection matrix [5] has been introduced Table 1 to perform the task.

$$h_m(t) h_m^T(t) c_m = C_m h_m(t) \quad (5)$$

$$M_m(t) c_m = C_m h_m(t) \quad (6)$$

Where C_m is the connection matrix

The Haar backward integration matrix [5] arises when integration is done backward in time from 1 to t where t<1. It can be expressed mathematically as

$$\int_1^t h_m(\tau)\, d\tau \ = \ S_m h_m(t) \tag{7}$$

With the help of backward integration matrix, integration can be easily converted into simple matrix multiplication.

These recursive formulations results in computationally costlier algorithms for higher order problems and it becomes even costlier when the high resolution is considered. Therefore a non recursive replacement of backward integration operational matrix has been introduced. Non recursive matrices result in computationally efficient algorithms, with respect to execution time and stack-and-memory overflows in computer implementations, as compared to corresponding recursive formulations. The non recursive Haar backward integration matrix is derived in [7] with the help of block pulse function as

$$S_m \ = h_m \{Q_{bm} - (\tfrac{1}{m})\} h_m^{-1} \tag{8}$$

Where,

Q_{bm} is integration matrix for block pulse function.

S_m is backward Haar integration matrix of order m.

h_m is Haar matrix for order m.

For further details, one can refer to [7].

In the next section, a method is proposed, based on non-recursive backward integral operational matrix for solving linear optimal control problems of high order. The proposed method is capable of optimizing - time-varying linear systems of any order efficiently; the proposed method is applied to solve finite horizon LQR problems with final state control. Computational efficiency of the proposed method is established with the help of comparison on computation-time at different resolutions.

OPTIMAL CONTROL OF LINEAR TIME VARYING SYSTEMS VIA HAAR WAVELET

Consider a general LQR linear time varying system with performance index J

$\dot{x}(t) = A\, x(t) + Bu(t)$ (9)

$$J = \int_0^{t_f} [X^T(t) Q(t) X(t) \ + \ u^T(t) R(t) u(t)] dt \tag{10}$$

Where A(t) is n x n matrix, B(t) is n x q matrix, Q (t) is a positive semi definite n X n matrix and R(t) is an r X r positive definite matrix.

For the solution of optimal control problem using Haar wavelets, the time interval $[0, t_f)$ is normalized to [0, 1], therefore the normalized performance index can be written as

$$J = t_f \int_0^1 [X^T(\tau) Q(t) X(\tau) + u^T(\tau) R(t) u(\tau)] d\tau \tag{11}$$

The adjoint state λ (τ), an n-vector, satisfies the following canonical equation:

$$\begin{bmatrix} \dot{x}(\tau) \\ \dot{\lambda}(\tau) \end{bmatrix} = t_f F(\tau) \begin{bmatrix} X(\tau) \\ \lambda(\tau) \end{bmatrix} \quad (12)$$

Where

$$F(\tau) = \begin{bmatrix} A(\tau) & -B(\tau) R^{-1}(\tau) B^T(\tau) \\ -Q(\tau) & -A^T(\tau) \end{bmatrix} \quad (13)$$

The initial condition for x(0) and λ(1)=0 (14)

Condition (14) is known as transversality condition. Let Φ(1,τ) be the state transition matrix of (12) which satisfies

$$\dot{\Phi}(1,\tau) = -t_f \Phi(1,\tau) F(\tau),\ \Phi(1,1)=I \quad (15)$$

Decomposing Φ(1,τ) into the following form:

$$\Phi(1,\tau) = \begin{bmatrix} \Phi_{11}(1,\tau) & \Phi_{12}(1,\tau) \\ \Phi_{21}(1,\tau) & \Phi_{22}(1,\tau) \end{bmatrix} \quad (16)$$

The transversality condition must hold leading to

$$\lambda(\tau) = -\Phi_{22}^{-1}(1,\tau)\Phi_{21}(1,\tau)x(\tau) \quad (17)$$

The time varying optimal feedback gain is

$$K(\tau) = -R^{-1}(\tau)B^T(\tau)\Phi 22^{-1}(1,\tau)\Phi_{21}(1,\tau) \quad (18)$$

Performing backward integration on (15)

$$\Phi(1,\tau) - I = -t_f \int_1^{\tau} \Phi(1,\tau) F(\tau) d\tau \quad (19)$$

Where

Φ(1, 1)=I

The Haar expansion of Φ(1,τ) can be expressed as

$$\Phi(1,\tau) = \begin{bmatrix} \Psi_{11}^T & \Psi_{12}^T & \cdots & \Psi_{1,2n}^T \\ \Psi_{21}^T & \Psi_{22}^T & \cdots & \Psi_{2,2n}^T \\ \vdots & \vdots & \vdots & \vdots \\ \Psi_{2n,1}^T & \Psi_{2n,2}^T & \cdots & \Psi_{2n,2n}^T \end{bmatrix} h_m(\tau) \quad (20)$$

Haar series approximation of $t_f F(\tau)$ is $t_f F(\tau) = \begin{bmatrix} ¥_{11}^T & ¥_{12}^T & \cdots & ¥_{1,2n}^T \\ ¥_{21}^T & ¥_{22}^T & \cdots & ¥_{2,2n}^T \\ \vdots & \vdots & \vdots & \vdots \\ ¥_{2n,1}^T & ¥_{2n,2}^T & \cdots & ¥_{2n,2n}^T \end{bmatrix} h_m(\tau)$ (21)

The order of matrices (20) and (21) is 2n X 2n and the Ψ_{ij} and $¥_{ij}$ are m vector

Substituting the values of $t_f \Phi(1,\tau) F(\tau)$ and Φ(1,τ) in (19) and compare the coefficient of h_m , (19) becomes [9]

$$\begin{bmatrix} \Psi_{11}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{j1} S_m & \Psi_{12}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{j2} S_m & \cdots & \Psi_{1,2n}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{j,2n} S_m \\ \Psi_{21}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{j2} S_m & \Psi_{22}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{j2} S_m & \cdots & \Psi_{2,2n}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{j2} S_m \\ \vdots & \vdots \quad \vdots & & \vdots \\ \Psi_{2n.1}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{ji} S_m & \Psi_{2n,2}^T \sum_{j=1}^{2n} \Psi_{1j}^T C_{ji} S_m & \cdots & \Psi_{2n,2n}^T + \sum_{j=1}^{2n} \Psi_{1j}^T C_{ji} S_m \end{bmatrix}$$

$$= \begin{bmatrix} 1\ 0\ 0 \ldots 0 & 0\ 0\ 0 \ldots 0 & \cdots & 0\ 0\ 0 \ldots 0 \\ 0\ 0\ 0 \ldots 0 & 1\ 0\ 0 \ldots 0 & & 0\ 0\ 0 \ldots 0 \\ \vdots & \ddots & & \vdots \\ 0\ 0\ 0 \ldots 0 & 0\ 0\ 0 \ldots 0 & \cdots & 1\ 0\ 0 \ldots 0 \end{bmatrix} \tag{22}$$

Where C_{ij} represent connection matrix for F_{ij} element

The left hand side of (22) can be expressed after rearrangement as

$$\begin{bmatrix} 1 + \sum_{j=1}^{2n} C_{j1} S_m & \sum_{j=1}^{2n} C_{j2} S_m & \cdots & \sum_{j=1}^{2n} C_{j,2n} S_m \\ \sum_{j=1}^{2n} C_{j2} S_m & 1 + \sum_{j=1}^{2n} C_{j2} S_m & \cdots & \sum_{j=1}^{2n} C_{j2} S_m \\ \vdots & \vdots \quad \vdots & & \vdots \\ \sum_{j=1}^{2n} C_{ji} S_m & \sum_{j=1}^{2n} C_{ji} S_m & \cdots & 1 + \sum_{j=1}^{2n} C_{ji} S_m \end{bmatrix} \mathrm{x} \begin{bmatrix} \Psi_{11}^T & \Psi_{12}^T & \cdots & \Psi_{1n}^T \\ \Psi_{21}^T & \Psi_{22}^T & & \Psi_{2n}^T \\ \vdots & & \ddots & \vdots \\ \Psi_{n1}^T & \Psi_{n2}^T & \cdots & \Psi_{nn}^T \end{bmatrix} \tag{23}$$

Applying linear algebra and rearranging matrices (23) can be expressed as

$$\left[\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{2n} \end{bmatrix} + I \right] \mathrm{x}\ [\Psi_i] = \begin{bmatrix} 1\ 0\ 0 \ldots 0 & 0\ 0\ 0 \ldots 0 & \cdots & 0\ 0\ 0 \ldots 0 \\ 0\ 0\ 0 \ldots 0 & 1\ 0\ 0 \ldots 0 & & 0\ 0\ 0 \ldots 0 \\ \vdots & \ddots & & \vdots \\ 0\ 0\ 0 \ldots 0 & 0\ 0\ 0 \ldots 0 & \cdots & 1\ 0\ 0 \ldots 0 \end{bmatrix} \tag{24}$$

Where

$U_i = [S_m^T \Omega_{1i}^T,\ S_m^T \Omega_{2i}^T, S_m^T \Omega_{3i}^T \ldots\ldots\ldots\ldots S_m^T \Omega_{2n,i}^T]$

$\Psi_i = [\Psi_{i1}^T, \Psi_{i1}^T, \Psi_{i1}^T \ldots \ldots \ldots \ldots . . \Psi_{i1}^T]^T$

The reverse integration matrix S can be calculated using (11)

Let

$$V = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{2n} \end{bmatrix} + I \tag{25}$$

Then

$$\Psi_i = V^{-1} z_i \tag{26}$$

$z_1 = [1\ 0\ 0\ \ldots\ 0 : 0\ 0\ 0\ \ldots\ 0 : \ldots . : 0\ 0\ 0\ \ldots\ 0]^T$

$z_2 = [0\ 0\ 0\ \ldots\ 0 : 1\ 0\ 0\ \ldots\ 0 : \ldots . : 0\ 0\ 0\ \ldots\ 0]^T$

$\vdots$

$z_{2n} = [0\ 0\ 0\ \ldots\ 0 : \ 0\ 0\ 0\ \ldots\ 0 : \ \ldots . : \ 1\ 0\ 0\ \ldots\ 0]^T$

$\Phi(1,\tau)$ can be calculated using (20). $\Phi_{21}(1,\tau)$ and $\Phi_{22}(1,\tau)$ can be calculated using (16).

We have

$$\begin{bmatrix} x(1) \\ \lambda(1) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(1,\tau) & \Phi_{12}(1,\tau) \\ \Phi_{21}(1,\tau) & \Phi_{22}(1,\tau) \end{bmatrix} \begin{bmatrix} x(\tau) \\ \lambda(\tau) \end{bmatrix} \tag{27}$$

The transversality condition () must hold and leading to

$$\lambda(\tau) = \Phi_{22}^{-1}(1,\tau)\Phi_{21}(1,\tau)x(\tau) \tag{28}$$

The well known optimal control law is

$$u(\tau) = -R^{-1}(\tau)B^T(\tau)\lambda(\tau) \tag{29}$$

The feedback control law is

$$u(\tau) = -K(\tau)x(\tau) \tag{30}$$

from (28-30) the equation of K given in (18) can be derived.

A third order time varying problem, whose analytical solution is not possible, has been explained using modified algorithm in next section.

NUMERICAL EXAMPLE

Consider the state space equations as

$$\dot{x}(t) = \begin{bmatrix} .01t & .01 & 0 \\ 0 & .01t & 0 \\ 0 & 0 & .01t^2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .01 \\ .01 \end{bmatrix} u(t) \tag{31}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x(t_f) \text{ unspecified} \tag{32}$$

The performance index of linear LQR problem is given as

$$J = \int_0^{t_f} [X^T(t)Q(t)X(t) + u^T(t)R(t)u(t)]dt \tag{33}$$

Where

$$Q(t) = \begin{bmatrix} .01t & 0 & 0 \\ 0 & .01t & 0 \\ 0 & 0 & .01t \end{bmatrix} \tag{34}$$

$R(t) = .01\exp(-.01t)$, $t \in [0,5)$ (35)

The time-varying optimal feedback gain $K(t)$, $t \in [0, 5)$, is required to be found [7]

Comparing with (11)

$$A(t) = \begin{bmatrix} .01t & .01 & 0 \\ 0 & .01t & 0 \\ 0 & 0 & .01t^2 \end{bmatrix}, B(t) = \begin{bmatrix} 0 \\ .01 \\ .01 \end{bmatrix} \tag{36}$$

All calculation here has been done considering resolution m=8

Step 1

Calculating F(τ) from (13)

$$F(\tau) = \begin{bmatrix} .25\tau & .05 & 0 & 0 & 0 & 0 \\ 0 & .25\tau & 0 & 0 & -.05e^{.05\tau} & -.05e^{.05\tau} \\ 0 & 0 & 1.25\tau^2 & 0 & -.05e^{.05\tau} & -.05e^{.05\tau} \\ -.25\tau & 0 & 0 & -.25\tau & 0 & 0 \\ 0 & -.25\tau & 0 & -.05 & -.25\tau & 0 \\ 0 & 0 & -.25\tau & 0 & 0 & -1.25\tau^2 \end{bmatrix}$$

Step 2

Calculating S from table I

$$S = \begin{pmatrix} -1/2 & 1/4 & 1/16 & 1/16 & 1/64 & 1/64 & 1/64 & 1/64 \\ -1/4 & 0 & 1/16 & -1/16 & 1/64 & 1/64 & -1/64 & -1/64 \\ -1/8 & -1/8 & 0 & 0 & 1/32 & -1/32 & 0 & 0 \\ -1/8 & 1/8 & 0 & 0 & 0 & 0 & 1/32 & -1/32 \\ -1/16 & -1/16 & -1/16 & 0 & 0 & 0 & 0 & 0 \\ -1/16 & -1/16 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ -1/16 & 1/16 & 0 & -1/16 & 0 & 0 & 0 & 0 \\ -1/16 & 1/16 & 0 & 1/16 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3

Calculating V from (25)

$$V = \begin{pmatrix} 1.0873 & -0.0666 & -0.0079 & -0.0279 & \ldots\ldots\ldots\ldots 0 \\ 0.0338 & 0.9870 & -0.0079 & 0.0069 & \ldots\ldots\ldots\ldots 0 \\ 0.0087 & 0.0072 & 0.9969 & -0.0001 & \ldots\ldots\ldots\ldots 0 \\ 0.0246 & -0.0246 & -0.0000 & 0.9962 & \ldots\ldots\ldots\ldots 0 \\ :: & :: & :: & & \\ 0 & 0 & 0 & 0 & \ldots\ldots\ldots 1.0044 \end{pmatrix}$$

Step 4

Calculating Ψ_i from (26)

Ψ_1= [1.0873 0.0338 0.0087…… -0.0001 0.0000]^T

Ψ_2= [0.0000 0.0000 0.0000 ……-0.0029 -0.0030]^T

Ψ_3= [0.0000 0.0000 0.0000 ……-0.0038 -0.0034]^T

Ψ_4= [-0.0832 -0.0314 -0.0079 …… 0.0000 0.0000]^T

Ψ_5= [0.0010 0.0006 0.0003 …… 0.0002 0.0001]^T

Ψ_6= [0.0000 0.0000 0.0000 ……-0.0026 -0.0534]^T

Step 5

Calculating $\Phi_{21}(1,\tau)$ and $\Phi_{22}(1,\tau)$ from (16)

$$\Phi_{21}(1,\tau)=\begin{bmatrix} -.1243 & 0.0000 & \dots\dots & 0.0000 \\ 0.0000 & -.1204 & \dots\dots & 0.0000 \\ \vdots & \vdots & \dots\dots & \vdots \\ 0.0000 & 0.0000 & \dots\dots & -.0147 \end{bmatrix}$$

$$\Phi_{22}(1,\tau)=\begin{bmatrix} 0.8834 & 0.0000 & \dots\dots & 0.0000 \\ 0.0000 & 0.8868 & \dots\dots & 0.0000 \\ \vdots & \vdots & \dots\dots & \vdots \\ 0.0000 & 0.0000 & \dots\dots & 0.9358 \end{bmatrix}$$

Step 6

Calculate optimal gain K from (18)

K_1= [0.0044 0.0035 0.0026 0.0018 0.0011 0.0006 0.0002 -0.0000]

K_2= [0.1403 0.1363 0.1274 0.1136 0.0951 0.0723 0.0456 0.0156]

K_3= [0.1784 0.1748 0.1648 0.1471 0.1215 0.0892 0.0529 0.0165]

Following the procedure given in previous section, the solution of time-varying optimal control problem has been obtained. The result obtained is shown graphically in Figure 1 which satisfies with the existing solution.

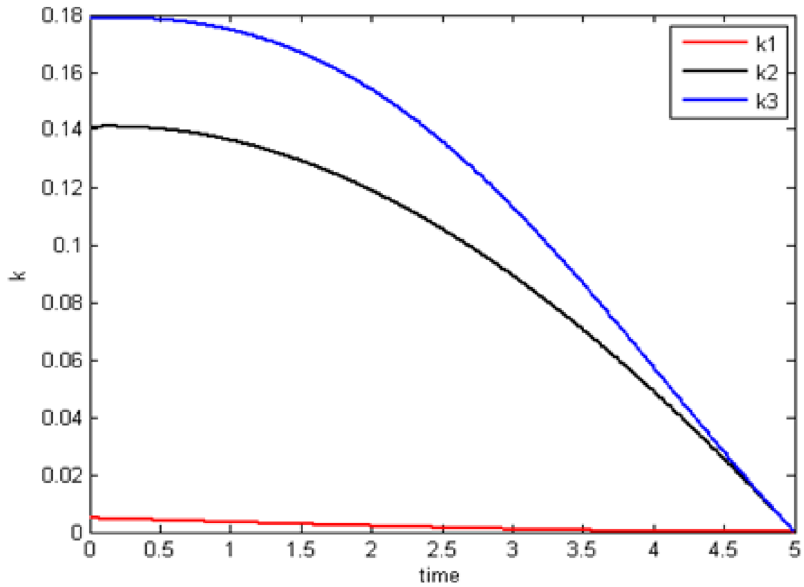

Figure 1: Optimal Gain K versus Time T

A graphical comparison between computational times involved in both the processes is shown in Figure 2, by examine the graphs it is confirmed that modified method is computationally much efficient.

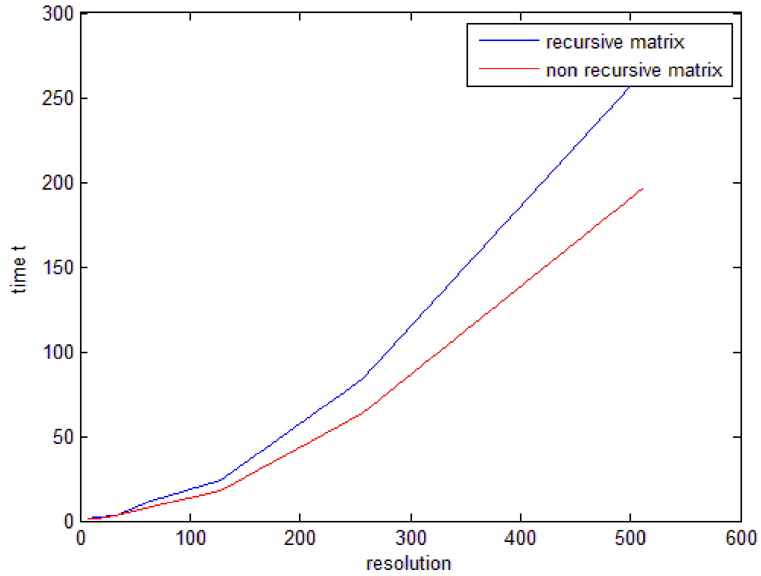

Figure 2: Time Consumption T versus Resolution M

CONCLUSIONS

Optimal control of a third order LTV system is successfully accomplished using modified Haar operational approach.Computational efficiency achieved using non-recursive Haar backward integral operator matrix in place of recursive one is demonstrated using MATLAB. The results obtained are shown to agree well with respective reported results [7,9]. The convergence is expected to be better at higher resolutions. Future scope lies in achieving optimal control using other non-recursive Haar operational matrices.

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